THE ASCENDING CHAIN CONDITION FOR LOG CANONICAL THRESHOLDS ON L.C.I. VARIETIES

TOMMASO DE FERNEX AND MIRCEA MUSTAŢĂ

Abstract. Shokurov's ACC Conjecture [Sho] says that the set of all log canonical thresholds on varieties of bounded dimension satisfies the Ascending Chain Condition. This conjecture was proved for log canonical thresholds on smooth varieties in [EM1]. Here we use this result and inversion of adjunction to establish the conjecture for locally complete intersection varieties.

1. Introduction

Let k be an algebraically closed field of characteristic zero. Log canonical varieties are varieties with mild singularities that provide the most general context for the Minimal Model Program. More generally, one puts the log canonicity condition on pairs $(X, \mathfrak{b}^q)$, where $\mathfrak{b}$ is a nonzero ideal on X (most of the times, it is the ideal of an effective Cartier divisor), and q is a nonnegative real number. Given a log canonical variety X over k, and an ideal sheaf $\mathfrak{b}$ on X with $(0) \neq \mathfrak{b} \neq \mathcal{O}_X$, one defines the log canonical threshold $\mathrm{lct}(X, \mathfrak{b})$ of the pair $(X, \mathfrak{b})$. This is the largest q such that the pair $(X, \mathfrak{b}^q)$ is log canonical. The log canonical threshold is a fundamental invariant in birational geometry, see for example [Kol2], [EM2], or Chap. 9 in [Laz].

Shokurov's ACC Conjecture [Sho] says that the set of all log canonical thresholds on varieties of bounded dimension satisfies the Ascending Chain Condition, that is, it contains no infinite strictly increasing sequences. This conjecture attracted considerable interest, due to its implications for the Termination of Flips Conjecture (see [Bir] for a result in this direction). The first unconditional results on sequences of log canonical thresholds on smooth varieties of arbitrary dimension have been obtained in [dFM], and they were subsequently reproved and strengthened in [Kol1]. The ACC Conjecture was proved in [EM1] for smooth ambient varieties, by reducing it to a key special case covered in [Kol1]. In this note we extend this result to the case when the ambient varieties are locally complete intersection (l.c.i., for short).

Theorem 1.1. *For every $n \geq 1$, the set*

$$\{\mathrm{lct}(X, \mathfrak{b}) \mid X \text{ is l.c.i. and log canonical}, (0) \neq \mathfrak{b} \neq \mathcal{O}_X, \dim(X) \leq n\}$$

satisfies the Ascending Chain Condition.

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In fact, we deal with a more general version of log canonical thresholds. Given a variety X and an ideal sheaf $\mathfrak{a}$ on X such that the pair $(X, \mathfrak{a})$ is log canonical, for every ideal sheaf $\mathfrak{b}$ on X with $(0) \neq \mathfrak{b} \neq \mathcal{O}_X$, we define the *mixed log canonical threshold* $\mathrm{lct}_{\mathfrak{a}}(X, \mathfrak{b})$ to be the largest q such that the pair $(X, \mathfrak{a} \cdot \mathfrak{b}^q)$ is log canonical. We will prove the following strengthening of the above theorem.

Theorem 1.2. *For every $n \geq 1$, the set*

$$\{\mathrm{lct}_{\mathfrak{a}}(X, \mathfrak{b}) \mid X \textit{ is l.c.i.}, (X, \mathfrak{a}) \textit{ is log canonical}, (0) \neq \mathfrak{b} \neq \mathcal{O}_X, \dim(X) \leq n\}$$

satisfies the Ascending Chain Condition.

A key point for the proof of Theorem 1.2 is that by Inversion of Adjunction we can express every number of the form $\mathrm{lct}_{\mathfrak{a}}(X, \mathfrak{b})$, with X locally complete intersection, as a similar invariant on a smooth variety (this is why it is important to work with mixed log canonical thresholds). Furthermore, we show that if X is an l.c.i. log canonical variety, then $\dim_k T_x X \leq 2\dim(X)$ for every $x \in X$. As a consequence, the above reduction to the smooth case keeps the dimension of the ambient variety bounded. After localizing at a suitable point, we reduce ourselves to the case when the ambient space is $\mathrm{Spec}\, k[\![x_1, \ldots, x_{2n}]\!]$. In this case we use the framework from [dFM] (or equivalently, from [Kol1]) to reduce the ACC statement to the case of usual log canonical thresholds on smooth varieties, that was treated in [EM1].

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2. Mixed log canonical thresholds

In this section we discuss mixed log canonical thresholds, and show how to reduce Theorem 1.2 to the case of ambient smooth varieties. We start by fixing the setup.

Let k be an algebraically closed field of characteristic zero. In what follows X will be either a variety over k or $\mathrm{Spec}\, k[\![x_1, \ldots, x_n]\!]$. For the basic facts about log canonical pairs in the setting of algebraic varieties, see [Kol2] or Chap. 9 in [Laz], while for the case of the spectrum of a formal power series ring we refer to [dFM]. The key point is that by [Tem], log resolutions exist also in the latter case, and therefore the usual theory of log canonical pairs carries through.

Suppose that X is as above, and $\mathfrak{a}$, $\mathfrak{b}$ are coherent nonzero sheaves of ideals on X such that the pair $(X, \mathfrak{a})$ is log canonical. In particular, this implies that X is normal and $\mathbf{Q}$-Gorenstein. We also assume that $\mathfrak{b} \neq \mathcal{O}_X$. The *mixed log canonical threshold* of $\mathfrak{b}$ with respect to $(X, \mathfrak{a})$ is

$$\mathrm{lct}_{\mathfrak{a}}(X, \mathfrak{b}) := \sup\{q \geq 0 \mid (X, \mathfrak{a} \cdot \mathfrak{b}^q) \text{ is log canonical}\}.$$

The fact that log canonicity can be checked on a log resolution allows us to describe this invariant in terms of such a resolution. Suppose that $\pi\colon Y \longrightarrow X$ is a log resolution

of $\mathfrak{a}\cdot\mathfrak{b}$, and write $\mathfrak{a}\cdot\mathcal{O}_Y=\mathcal{O}(-\sum_i a_iE_i)$, $\mathfrak{b}\cdot\mathcal{O}_Y=\mathcal{O}(-\sum_i b_iE_i)$, and $K_{Y/X}=\sum_i k_iE_i$. It follows from the characterization of log canonicity in terms of a log resolution that

$$\mathrm{lct}_{\mathfrak{a}}(X,\mathfrak{b})=\min\left\{\frac{k_i+1-a_i}{b_i}\mid b_i>0\right\}. \tag{1}$$

We see from the above formula that the mixed log canonical threshold is a rational number. Note also that it is zero if and only if there is i such that $k_i+1=a_i$ and $b_i>0$ (in other words, $(X,\mathfrak{a})$ is not klt, and there is a non-klt center contained in the support of $\mathfrak{b}$).

Remark 2.1. When $\mathfrak{a}=\mathcal{O}_X$, the mixed log canonical threshold $\mathrm{lct}_{\mathcal{O}_X}(X,\mathfrak{b})$ is nothing else than the usual log canonical threshold $\mathrm{lct}(X,\mathfrak{b})$, which we sometimes simply denote by $\mathrm{lct}(\mathfrak{b})$, when there is no ambiguity regarding the ambient scheme.

Remark 2.2. It follows from the description in terms of a log resolution that if $X=U_1\cup\ldots\cup U_r$, with U_i open, then

$$\mathrm{lct}_{\mathfrak{a}}(X,\mathfrak{b})=\min_i \mathrm{lct}_{\mathfrak{a}_i}(U_i,\mathfrak{b}_i),$$

where $\mathfrak{a}_i=\mathfrak{a}|_{U_i}$ and $\mathfrak{b}_i=\mathfrak{b}|_{U_i}$.

Remark 2.3. If $\mathfrak{a}$ and $\mathfrak{b}$ are as above, and $c=\mathrm{lct}_{\mathfrak{a}}(X,\mathfrak{b})$, then $\mathrm{lct}(X,\mathfrak{a}\cdot\mathfrak{b}^c)=1$. Indeed, by assumption the pair $(X,\mathfrak{a}\cdot\mathfrak{b}^c)$ is log canonical, and for every $\alpha>1$ the pair $(X,(\mathfrak{a}\cdot\mathfrak{b}^c)^{\alpha})$ is not log canonical since $(X,\mathfrak{a}\cdot\mathfrak{b}^{c\alpha})$ is not.

Remark 2.4. Suppose that X is a smooth variety over k, and $\mathfrak{a}$, $\mathfrak{b}$ are nonzero ideals on X, with $(X,\mathfrak{a})$ log canonical and $\mathfrak{b}\neq\mathcal{O}_X$. If $\mathrm{lct}_{\mathfrak{a}}(X,\mathfrak{b})=c$, then there is a (closed) point $x\in X$ such that $\mathrm{lct}_{\mathfrak{a}'}(X',\mathfrak{b}')=c$, where $X'=\mathrm{Spec}(\widehat{\mathcal{O}_{X,x}})$, and $\mathfrak{a}'=\mathfrak{a}\cdot\mathcal{O}_{X'}$, $\mathfrak{b}'=\mathfrak{b}\cdot\mathcal{O}_{X'}$. Indeed, if $\pi\colon Y\longrightarrow X$ is a log resolution of $(X,\mathfrak{a}\cdot\mathfrak{b})$, and if E_i is one of the divisors on Y for which the minimum in (1) is achieved, then it is enough to take $x\in\pi(E_i)$ (note that $Y\times_X X'$ gives a log resolution of $(X',\mathfrak{a}'\cdot\mathfrak{b}')$). In this way we are reduced to considering ideals in $k[\![x_1,\ldots,x_n]\!]$, where $n=\dim(X)$.

The following application of Inversion of Adjunction is a key tool in our study, as it allows us to replace mixed log canonical thresholds on locally complete intersection varieties with the similar type of invariants on ambient smooth varieties. In this result we assume that our schemes are of finite type over k.

Proposition 2.5. *Let A be a smooth irreducible variety over k, and $X\subset A$ a closed subvariety of pure codimension e, that is normal and locally a complete intersection. Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are ideals on A, with $\mathfrak{b}\neq\mathcal{O}_A$, and such that X is not contained in the union of the zero-loci of $\mathfrak{a}$ and $\mathfrak{b}$.*

i) *The pair $(X,\mathfrak{a}|_X)$ is log canonical if and only if for some open neighborhood U of X, the pair $(U,\mathfrak{a}\cdot\mathfrak{p}^e|_U)$ is log canonical, where $\mathfrak{p}$ is the ideal defining X in A.*
ii) *If $(X,\mathfrak{a}|_X)$ is log canonical, and if X intersects the zero-locus of $\mathfrak{b}$, then for some open neighborhood V of X we have*

$$\mathrm{lct}_{\mathfrak{a}|_X}(X,\mathfrak{b}|_X)=\mathrm{lct}_{\mathfrak{a}|_V\cdot\mathfrak{p}^e|_V}(V,\mathfrak{b}|_V).$$

Proof. Both assertions follow from Inversion of Adjunction (see Corollary 3.2 in [EM3]), as this says that for every nonnegative q, the pair $(X, (\mathfrak{a}\cdot\mathfrak{b}^q)|_X)$ is log canonical if and only if the pair $(A, \mathfrak{a}\cdot\mathfrak{b}^q\cdot\mathfrak{p}^e)$ is log canonical in some neighborhood of X. $\square$

The next proposition allows us to control the dimension of the smooth variety, when replacing a mixed log canonical threshold on an l.c.i. variety by one on a smooth variety. We keep the assumption that X is of finite type over k. Given a closed point $x\in X$, we denote by T_xX the Zariski tangent space of X at x.

Proposition 2.6. *Let X be a locally complete intersection variety. If X is log canonical, then $\dim_k T_xX \leq 2\dim(X)$ for every $x\in X$.*

Proof. Fix $x\in X$, and let $N = \dim\, T_xX$. After possibly replacing X by an open neighborhood of x, we may assume that we have a closed embedding of X in a smooth irreducible variety A, of pure codimension e, with $\dim(A) = N$. If $X = A$, then $N = \dim(X)$ and we are done.

Suppose now that $e\geq 1$. Since X is locally a complete intersection, it follows from Inversion of Adjunction (see Corollary 3.2 in [EM3]) that the pair $(A, \mathfrak{p}^e)$ is log canonical, where $\mathfrak{p}$ is the ideal of X in A. In particular, if E is the exceptional divisor of the blow-up A' of A at p, and ord_E is the corresponding valuation, then we have

$$N = 1 + \mathrm{ord}_E(K_{A'/A}) \geq e\cdot \mathrm{ord}_E(\mathfrak{p}) \geq 2e = 2(N-\dim(X)).$$

This gives $N\leq 2\dim(X)$. $\square$

Corollary 2.7. *In order to prove Theorem 1.2, it is enough to show that for every $N\geq 1$, the set*

$$\{\mathrm{lct}_{\mathfrak{a}}(W_N, \mathfrak{b}) \mid (W_N, \mathfrak{a}) \textit{ is log canonical}, \ (0)\neq \mathfrak{b}\neq \mathcal{O}_{W_N}\},$$

where $W_N = \mathrm{Spec}\, k[\![x_1,\ldots,x_N]\!]$, satisfies the Ascending Chain Condition.

Proof. Let us denote by $\mathcal{M}_N$ the set that appears in the statement. If each $\mathcal{M}_N$ satisfies ACC, then it is clear that in order to prove Theorem 1.2 it is enough to show

$$\{\mathrm{lct}_{\mathfrak{a}}(X,\mathfrak{b}) \mid X \text{ is l.c.i., } (X,\mathfrak{a}) \text{ is log canonical, } \dim(X)\leq n,\ (0)\neq\mathfrak{b}\neq\mathcal{O}_X\} \subseteq \bigcup_{i=1}^{2n}\mathcal{M}_i.$$

Suppose that $(X,\mathfrak{a})$ is log canonical, with X locally a complete intersection of dimension $\leq n$, and let $c = \mathrm{lct}_{\mathfrak{a}}(X,\mathfrak{b})$. If $x\in X$ is chosen as in Remark 2.4, then for every open neighborhood U of x we have $\mathrm{lct}_{\mathfrak{a}|_U}(U, \mathfrak{b}|_U) = c$. Since X is log canonical, it follows from Proposition 2.6 that $\dim_k T_xX\leq 2n$. After replacing X by a suitable neighborhood of x, we may assume that we have an embedding $X\hookrightarrow A$, where A is a smooth variety of dimension $m\leq 2n$. Proposition 2.5 implies that after possibly replacing A by a neighborhood of X, we have $c = \mathrm{lct}_{\mathfrak{a}_1\cdot\mathfrak{p}^e}(A, \mathfrak{b}_1)$, where $\mathfrak{p}$ is the ideal defining X in A, e is the codimension of X in A, and $\mathfrak{a}_1$ and $\mathfrak{b}_1$ are ideals in A whose restrictions to X give, respectively, $\mathfrak{a}$ and $\mathfrak{b}$. We conclude by replacing A with $\mathrm{Spec}\, k[\![x_1,\ldots,x_m]\!]$, as in Remark 2.4. $\square$

3. The Ascending Chain Condition

The proof of Theorem 1.2 uses a construction from [dFM], whose general properties we shall review first. Equivalently, one can replace this construction by the one considered in [Kol1], which does not rely on ultrafilters.

It is shown in [dFM] that we can find an algebraically closed extension K of k, and a way to associate to every sequence of ideals $\mathfrak{a}_m \subseteq k[\![x_1, \ldots, x_n]\!]$, an ideal $\widetilde{\mathfrak{a}}$ in $K[\![x_1, \ldots, x_n]\!]$, that we shall call *the limit ideal* of the sequence $(\mathfrak{a}_m)_m$. This construction satisfies the following properties:

i) If $\mathfrak{a}_m \neq (0)$ for every m, then $\widetilde{\mathfrak{a}} \neq (0)$.
ii) For every sequences of ideals $(\mathfrak{a}_m)_m$ and $(\mathfrak{b}_m)_m$ in $k[\![x_1, \ldots, x_n]\!]$, with limit ideals $\widetilde{\mathfrak{a}}$ and $\widetilde{\mathfrak{b}}$ in $K[\![x_1, \ldots, x_n]\!]$, the limit ideals of the sequences $(\mathfrak{a}_m \cdot \mathfrak{b}_m)_m$ and $(\mathfrak{a}_m + \mathfrak{b}_m)_m$ are, respectively, $\widetilde{\mathfrak{a}} \cdot \widetilde{\mathfrak{b}}$ and $\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{b}}$.
iii) If $\mathfrak{a}_m = \mathfrak{a}$ for every m, then $\widetilde{\mathfrak{a}} = \mathfrak{a} \cdot K[\![x_1, \ldots, x_n]\!]$.
iv) If there is an integer d such that each $\mathfrak{a}_m$ can be generated by polynomials of degree $\leq d$, then the same holds for $\widetilde{\mathfrak{a}}$, and there are infinitely many m such that $\mathrm{lct}(\mathfrak{a}_m) = \mathrm{lct}(\widetilde{\mathfrak{a}})$.

The way one applies this construction in [dFM] is the following. Suppose that $(\mathfrak{a}_m)_m$ is a sequence of proper nonzero ideals in $k[\![x_1, \ldots, x_n]\!]$, and let $c_m = \mathrm{lct}(\mathfrak{a}_m)$. By properties ii) and iii) above, if we denote by $\mathfrak{n}$ the maximal ideal in $k[\![x_1, \ldots, x_n]\!]$, then the limit ideal $\widetilde{\mathfrak{n}}$ of the constant sequence $(\mathfrak{n})$ is the maximal ideal in $K[\![x_1, \ldots, x_n]\!]$, and moreover, for every $d \geq 1$ the limit ideal of the sequence $(\mathfrak{a}_m + \mathfrak{n}^d)_m$ is $\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{n}}^d$. Note also that from properties i) and ii) we deduce that $\widetilde{\mathfrak{a}}$ is a proper nonzero ideal in $K[\![x_1, \ldots, x_n]\!]$. Since each $\mathfrak{a}_m + \mathfrak{n}^d$ is generated in degree $\leq d$, it follows from iv) that we can find infinitely many m such that

$$\mathrm{lct}(\mathfrak{a}_m + \mathfrak{n}^d) = \mathrm{lct}(\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{n}}^d).$$

On the other hand, well-known properties of log canonical thresholds (see, for example, Corollary 2.11 in [dFM]) give for every m and d

$$0 \leq \mathrm{lct}(\mathfrak{a}_m + \mathfrak{n}^d) - \mathrm{lct}(\mathfrak{a}_m) \leq \frac{n}{d},$$

$$0 \leq \mathrm{lct}(\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{n}}^d) - \mathrm{lct}(\widetilde{\mathfrak{a}}) \leq \frac{n}{d}.$$

It follows that given any d, there are infinitely many m such that $|\,\mathrm{lct}(\widetilde{\mathfrak{a}}) - \mathrm{lct}(\mathfrak{a}_m)| \leq \frac{2n}{d}$. For example, one deduces from this that if $\lim_{m \longrightarrow \infty} \mathrm{lct}(\mathfrak{a}_m) = c$, then $\mathrm{lct}(\widetilde{\mathfrak{a}}) = c$. We refer to [dFM] for details.

For the proof of Theorem 1.2, we will need the following proposition. We point out that while the assertion in i) is an immediate consequence of the above formalism, the one in ii) is essentially a restatement of the result in [EM1] saying that the ACC Conjecture holds for log canonical thresholds on smooth varieties.

Proposition 3.1. *With the above notation, the following hold:*

i) *If* $\mathrm{lct}(\mathfrak{a}_m) \geq \tau$ *for every* $m \gg 0$*, then* $\mathrm{lct}(\widetilde{\mathfrak{a}}) \geq \tau$*.*
ii) *There are infinitely many* m *such that* $\mathrm{lct}(\mathfrak{a}_m) \geq \mathrm{lct}(\widetilde{\mathfrak{a}})$*.*

Proof. The first assertion follows immediately from the above discussion. Indeed, since $\mathrm{lct}(\widetilde{\mathfrak{a}}) = \lim_{d\longrightarrow\infty} \mathrm{lct}(\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{n}}^d)$, it is enough to show that $\mathrm{lct}(\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{n}}^d) \geq \tau$ for every d. This follows since given d, we can find $m \gg 0$ such that

$$\mathrm{lct}(\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{n}}^d) = \mathrm{lct}(\mathfrak{a}_m + \mathfrak{n}^d) \geq \mathrm{lct}(\mathfrak{a}_m) \geq \tau.$$

For ii), we use the fact that the sequence $(\mathrm{lct}(\mathfrak{a}_m))_m$ contains no strictly increasing infinite subsequences. This is a consequence of the fact that by Theorem 1.1 in [EM1], the set

$$\mathcal{T}_n := \{\mathrm{lct}(\mathfrak{a}) \mid \mathfrak{a} \subseteq k[\![x_1, \ldots, x_n]\!], \mathfrak{a} \neq (0), \mathrm{ord}(\mathfrak{a}) \geq 1\}$$

satisfies ACC. In fact, the result in *loc. cit.* concerns log canonical thresholds of principal ideals in $k[x_1, \ldots, x_n]$, but this implies our assertion by a well-known argument (see, for example, Proposition 3.6 and Corolary 3.8 in [dFM]).

In particular, we see that there is $\varepsilon > 0$ such that no $\mathrm{lct}(\mathfrak{a}_m)$ lies in the open interval $(\mathrm{lct}(\widetilde{\mathfrak{a}}) - \varepsilon, \mathrm{lct}(\widetilde{\mathfrak{a}}))$. Let d be such that $\frac{2n}{d} < \varepsilon$. We have seen that there are infinitely many m such that $|\,\mathrm{lct}(\widetilde{\mathfrak{a}}) - \mathrm{lct}(\mathfrak{a}_m)| < \frac{2n}{d}$, and by the choice of d, it follows that for all such m we have $\mathrm{lct}(\mathfrak{a}_m) \geq \mathrm{lct}(\widetilde{\mathfrak{a}})$. $\square$

We can now give the proof of our main result.

Proof of Theorem 1.2. In light of Corollary 2.7, it is enough to prove that for every $n \geq 1$, if we put $X = \mathrm{Spec}\, k[\![x_1, \ldots, x_n]\!]$, then the set

$$\mathcal{M}_n := \{\mathrm{lct}_{\mathfrak{a}}(X, \mathfrak{b}) \mid (X, \mathfrak{a}) \text{ is log canonical, } (0) \neq \mathfrak{b} \neq \mathcal{O}_X\}$$

satisfies ACC. Suppose that we have a strictly increasing sequence $(c_m)_m$, with $c_m = \mathrm{lct}_{\mathfrak{a}_m}(X, \mathfrak{b}_m)$. Let $c = \lim_{m\longrightarrow\infty} c_m$ (since we have $\mathrm{lct}_{\mathfrak{a}}(X, \mathfrak{b}) \leq \mathrm{lct}(X, \mathfrak{b}) \leq n$, this sequence is convergent).

We now apply the construction mentioned in the beginning of the section to get the algebraically closed extension K of k and, for each of the sequences $(\mathfrak{a}_m)_m$ and $(\mathfrak{b}_m)_m$, its limit ideal $\widetilde{\mathfrak{a}}$, respectively $\widetilde{\mathfrak{b}}$, in $K[\![x_1, \ldots, x_n]\!]$. We want to compare c with $\widetilde{c} := \mathrm{lct}_{\widetilde{\mathfrak{a}}}(\widetilde{X}, \widetilde{\mathfrak{b}})$, where $\widetilde{X} = \mathrm{Spec}\, K[\![x_1, \ldots, x_n]\!]$. Note that by assumption this mixed log canonical threshold is well-defined, since we have $\mathrm{lct}(\mathfrak{a}_m) \geq 1$ for every m, hence $\mathrm{lct}(\widetilde{\mathfrak{a}}) \geq 1$ by assertion i) in Proposition 3.1.

Consider first any positive integers p and q such that $p/q < c$. By assumption, we have $c_m > p/q$ for all $m \gg 0$. Therefore the pair $(X, \mathfrak{a}_m \cdot \mathfrak{b}_m^{p/q})$ is log canonical, hence $\mathrm{lct}(\mathfrak{a}_m^q \cdot \mathfrak{b}_m^p) \geq 1/q$ for all $m \gg 0$. As we have mentioned, the limit ideal in $K[\![x_1, \ldots, x_n]\!]$ of the sequence $(\mathfrak{a}_m^q \cdot \mathfrak{b}_m^p)_m$ is $\widetilde{\mathfrak{a}}^q \cdot \widetilde{\mathfrak{b}}^p$. Assertion i) in Proposition 3.1 implies that $\mathrm{lct}(\widetilde{\mathfrak{a}}^q \cdot \widetilde{\mathfrak{b}}^p) \geq 1/q$, hence $\widetilde{c} = \mathrm{lct}_{\widetilde{\mathfrak{a}}}(\widetilde{X}, \widetilde{\mathfrak{b}}) \geq p/q$. As this holds for every $p/q < c$, we conclude that $\widetilde{c} \geq c$.

On the other hand, since $\widetilde{c} \in \mathbf{Q}$, we may write $\widetilde{c} = \frac{r}{s}$, for positive integers r and s. It follows from Remark 2.3 that $\mathrm{lct}(\widetilde{\mathfrak{a}} \cdot \widetilde{\mathfrak{b}}^{r/s}) = 1$, hence $\mathrm{lct}(\widetilde{\mathfrak{a}}^s \cdot \widetilde{\mathfrak{b}}^r) = 1/s$. Since $\widetilde{\mathfrak{a}}^s \cdot \widetilde{\mathfrak{b}}^r$ is the limit ideal of the sequence $(\mathfrak{a}_m^s \cdot \mathfrak{b}_m^r)_m$, assertion ii) in Proposition 3.1 implies that there are infinitely many m such that $\mathrm{lct}(\mathfrak{a}_m^s \cdot \mathfrak{b}_m^r) \geq 1/s$, hence $\mathrm{lct}_{\mathfrak{a}_m}(X, \mathfrak{b}_m) \geq r/s$. For such m we have

$$\widetilde{c} \geq c > c_m \geq \frac{r}{s} = \widetilde{c},$$

a contradiction. This completes the proof of the theorem. $\square$

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Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 48112-0090, USA

E-mail address: `defernex@math.utah.edu`

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA

E-mail address: `mmustata@umich.edu`